A SUPERPROCESS WITH A DISAPPEARING SELF-INTERACTION

by

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Abstract: We start with a system of independent branching Brownian motions which, properly organised and normalised, generate a super Brownian motion in the high density limit. We introduce a weak interaction between the particles, that affects the diffusion but not the branching. The interaction is chosen in such a way that the infinite density limit is absolutely continuous with respect to the non-interacting system.

We find that, despite the fact that the interaction mechanism never completely disappears, the limiting superprocesses are *identical*. We study what actually happens to the interaction mechanism, or "ghost-process".

Résumé: Nous considerons une système de mouvements Browniens en branchement que après normalisation adéquate convergent dans la limite de densité infini vers une super mouvement Brownien. Nous introduisans une interaction faible entre les particules que affecté le mechanisme de diffusion mais pas celui du branchements. L'interaction est diviser de telle manière que que la limite lorsque la densité tend vers l'infinite est absolument continue par rapport à la loi du système sans interaction.

En fait bien que la trace de l'interaction persiste les processus limites sont identiques. Nous ètudians le dépérissement du niveau du mechanisme d'interaction, le "processus fantôme".

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1. INTRODUCTION

This paper is about a curiosity in the behaviour of systems of branching Brownian motions converging to a super Brownian motion. The project behind it started with an attempt to introduce into a system of branching Brownian motions a mild interaction, via the diffusion rather than the branching, that would ultimately lead to an infinite density measure valued limit that would be different from the regular super Brownian motion, and yet generate a measure on the space of continuous measure valued processes that was absolutely continuous with respect to that of super Brownian motion.

It has been known since Dawson (1978) that absolutely continuous changes of measure of this kind exist when the interaction between the particles occurs via the branching mechanism rather than via the motion of the particles. It has also been part of the folklore of superprocesses for probably almost as long that quite simple changes to the motion of the particles (such as adding a constant drift to each particle) lead to superprocesses that are mutually singular. Our aim was to find a very mild interaction, that *almost* disappeared in the limit, but still left enough of a trace that the ultimate process would have a nice Radon-Nikodym derivative with respect to the measure generated by super Brownian motion. Although we were unsuccessful in our search, the path along the way yielded some interesting insights.

To describe these, we require some notation.

(a) Background and notation. We start with a parameter $\mu > 0$ that will eventually become large, a probability measure m on $\Re^d$, and $K_\mu = O(\mu)$ points in $\Re^d$ independently distributed according to m. These random points, $x_1, \ldots, x_{K_\mu}$, are to be the initial positions of a system of particles. The K_μ particles perform a $d \times K_\mu$ dimensional diffusion until time $t = 1/\mu$. At this time each particle, independently of the others, either dies or splits into two, each event having probability $\frac{1}{2}$. The individual particles in the new population then continue moving during the time interval $[1/\mu, 2/\mu)$, and the pattern of alternating critical branching and spatial spreading continues until, with probability one, there are no particles left alive.

In order to describe the particle motions, we first need the family of multi-indices

$$(1.1) \qquad I := \big\{ \alpha = (\alpha_0, \ldots, \alpha_N) \colon \alpha_0 \in \mathcal{N}, \ \alpha_i \in \{1, 2\}, \ i \geq 1, \ N \geq 0 \big\}.$$

Define the "length" of α by $|\alpha| = N$, and set $\alpha|_i = (\alpha_0, \ldots, \alpha_i)$ and $\alpha - i = (\alpha_0, \ldots, \alpha_{|\alpha|-i})$. Furthermore, for any $t > 0$, write $\alpha \sim t$, if, and only if

$$(1.2) \qquad \frac{|\alpha|}{\mu} \leq t < \frac{1 + |\alpha|}{\mu}.$$

Now let W_t^α be independent (for different $\alpha \in I$) Brownian motions for $t \sim \alpha$, and let N^α be independent copies of a random variable taking the values 0 and 2 with equal probabilities. The N^α are assumed independent of W_t^α.

We also require a family of indicator variables, telling when particles are "alive" or not, according to the branching mechanism. To this end, for each $\alpha \in I$ with $1 \leq \alpha_0 \leq K_\mu$ define the random variables h^α recursively by setting

$$h^{\alpha_0} = 1, \quad \alpha_0 \in \{1, \dots, K_\mu\},$$
$$(1.3) \qquad\qquad h^\alpha = \tfrac{1}{2} N^{\alpha-1} h^{\alpha-1}.$$

Finally, append to the state space $\Re^d$ a cemetery state Λ, and adopt the convention that $\phi(\Lambda) \equiv 0$ for all functions $\phi: \Re^d \to \Re^k$ Now we can define a collection of processes X^α by setting

$$(1.4) \qquad\qquad X^\alpha(0) \;=\; \begin{cases} x_{\alpha_0} & \text{if } 1 \leq \alpha_0 \leq K_\mu \\ \Lambda & \text{otherwise} \end{cases}$$

and for times $t \in [k/\mu, (k+1)/\mu)$ by setting, for α with $|\alpha| = k$,

$$(1.5) \quad X^\alpha(t) \;=\; \lim_{s \uparrow \frac{k}{\mu}} X^{\alpha-1}(s) + h^\alpha \int_{\frac{k}{\mu}}^t W^\alpha(ds) + \mu^{-\gamma} \sum_{|\beta|=k, \beta \neq \alpha} h^\alpha h^\beta \int_{\frac{k}{\mu}}^t b(X_s^\alpha, X_s^\beta)\, ds.$$

if $1 \leq \alpha_0 \leq K_\mu$, and Λ otherwise. (If $k = 0$ replace the first term on the right by $X^\alpha(0)$.) Here $b: \Re^d \times \Re^d \to \Re^d$ is some nice function and $\gamma \geq 1$ is a scale parameter to be chosen later. The process of interest to us is the measure valued Markov process

$$X_t^\mu(A) \;:=\; \mu^{-1}\{\text{Number of particles in } A \text{ at time } t\}$$
$$(1.6) \qquad\qquad = \mu^{-1} \sum_{\alpha \sim t} 1_A(X_t^\alpha) h^\alpha,$$

where $A \in \mathcal{B}^d$, the Borel σ-algebra in $\Re^d$.

When $b \equiv 0$ (or $\gamma = \infty$) in (1.5), then well known results dating back to Watanabe (1968) give that, as $\mu \to \infty$, the sequence $\{X_t^\mu\}_{\mu \geq 1}$ converges weakly on $D([0, \infty), M_F(\Re^d))$, the Skorohod space of cadlag functions from $[0, \infty)$ to $M_F(\Re^d)$, the space of finite Radon measures on $\Re^d$ endowed with the topology of weak convergence. The limit process is known as super Brownian motion, and can be characterised, for example, as the unique solution of the following martingale problem:

For all $\phi \in C_b^2(\Re^d)$, the space of all bounded, continuous, $\Re$ valued functions on $\Re^d$ with continuous first and second order partial derivatives,

$$(1.7a) \qquad\qquad Z_t(\phi) \;=\; X_t(\phi) - m(\phi) - \int_0^t X_s(\tfrac{1}{2}\Delta\phi)\, ds,$$

is a continuous, square integrable martingale such that $Z_0 = 0$ and

$$(1.7b) \qquad\qquad \langle Z(\phi) \rangle_t \;=\; \int_0^t X_s(\phi^2)\, ds.$$

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(We have taken the obvious liberty here of denoting integration via $\int \phi(x)\, \nu(dx) = \nu(\phi)$ for a measure ν. Later, without further comment, we shall also write this as $\langle \phi, \nu \rangle$.)

When b is nice, for example bounded Lipschitz, and $\gamma = 1$, then Perkins (1993) (cf. Perkins 1992) has shown that weak convergence still goes through, although now the limit process is somewhat different, in that the right hand side of (1.7a) now contains the additional "drift" term

$$(1.8) \qquad -\int_0^t \int_{\Re^d} \int_{\Re^d} \nabla \phi(x) \cdot b(x,y)\, X_s(dx) X_s(dy)\, ds.$$

The martingale squared variation process (1.7b) remains unchanged.

It is not hard to see, and will follow from the calculations below, that the interaction intrinisic in (1.8) leads to a process that is singular with respect to regular super Brownian motion. Consequently, this is not the interaction that we seek.

(b) A milder interaction and Radon-Nikodym derivatives. In order to obtain a mild interaction, based on (1.5), that will give us what we seek, we start with the finite system.

Note that if we add a time variable to the indicators h^α and branching variable N^α by setting $h^\alpha(t) \equiv h^\alpha$ and $N^\alpha(t) \equiv N^\alpha$ for all $t \geq 0$, then on the time interval $(k/\mu, (k+1)/\mu]$ we can regard the entire collection

$$(1.9) \qquad \left\{ X^\alpha(t),\, h^\alpha(t),\, N^\alpha(t) \right\}_{\alpha \in I}$$

as a diffusion on $\Re^{2^k dK_\mu} \times \{0,1\}^{2^k K_\mu} \times \{0,2\}^{2^{k-1} K_\mu}$.

This being the case, we could now use Girsanov's theorem to calculate, in the finite case, what the Radon-Nikodym derivative is for the system with interaction (general γ and b) with respect to the system without interaction ($b = 0$). We could then go to the limit in such a way that we keep a nice, limiting, derivative.

It turns out that the derivative takes a particularly simple form, if we introduce just one more process. This is the process

$$(1.10) \qquad W^\mu((0,t] \times A) := \mu^{-1/2} \int_0^t \sum_{\alpha \sim s} h_s^\alpha \, 1_A(X_s^\alpha)\, W^\alpha(ds).$$

This process can be treated as an $\Re^d$-valued martingale measure in the sense of Walsh (1986), a signed, vector, measure valued process, or a (Schwartz) distribution valued process. For the last of these, let $\mathcal{S}_d$ be the class of rapidly decreasing functions on $\Re^d$, and $\mathcal{S}_d'$ the corresponding class of Schwartz distributions. Note that W^μ is a sort of "summary" of the random diffusion in the system. It depends on the X^α, and so will carry some information about the interaction. The normalisation in (1.10) is such that this process has, as $\mu \to \infty$, a nice (weak) limit as a continuous $\mathcal{S}_d'$-valued process. This limit is what we shall call, for reasons to become clear soon, the *ghost process*.

To make the forthcoming results easier to write, let $\widetilde{X}^\mu$ and $\widetilde{W}^\mu$ be versions of X^μ and W^μ when there is no interaction ($b = 0$). Then we shall prove the following:

THEOREM 1.1. *Let Q^μ denote the probability measure generated on $D([0,T], M_F(\Re^d) \times (S'_d)^{\otimes d})$ by the pair (X^μ, W^μ), and P^μ that generated by $(\widetilde{X}^\mu, \widetilde{W}^\mu)$. Then, if b is bounded Lipschitz on $\Re^{2d}$, the Radon-Nikodym derivative of Q^μ with respect to P^μ is given by $J^\mu(\widetilde{X}^\mu, \widetilde{W}^\mu)$, where*

$$
\begin{aligned}
\log\left(J^\mu(\widetilde{X}^\mu, \widetilde{W}^\mu)\right) := \ & \mu^{3/2-\gamma} \int_0^T \int_{\Re^{2d}} b(x,y)\, \widetilde{X}^\mu_s(dy)\widetilde{W}^\mu(dx,ds) \\
& - \tfrac{1}{2}\mu^{3-2\gamma} \int_0^T \int_{\Re^{3d}} \langle b(x,y), b(x,z)\rangle\, \widetilde{X}^\mu_s(dx)\widetilde{X}^\mu_s(dy)\widetilde{X}^\mu_s(dz)\, ds \\
& - \mu^{1/2-\gamma} \int_0^T \int_{\Re^d} \langle b(x,x), \widetilde{W}^\mu(dx,ds)\rangle \\
& + \mu^{2-2\gamma} \int_0^T \int_{\Re^{2d}} \langle b(x,y), b(x,x)\rangle\, \widetilde{X}^\mu_s(dx)\widetilde{X}^\mu_s(dy)\, ds \\
& - \tfrac{1}{2}\mu^{1-2\gamma} \int_0^T \int_{\Re^d} \|b(x,x)\|^2\, \widetilde{X}^\mu_s(dx)\, ds.
\end{aligned}
$$

(1.11)

(Here angle brackets represent vector inner products, and $\|\cdot\|$ the usual Euclidean norm.)

Looking carefully at the exponents of μ in (1.11), and keeping in mind that both $\widetilde{X}^\mu$ and $\widetilde{W}^\mu$ go to well defined limits as $\mu \to \infty$, it is now "clear" what happens as different choices of γ are made. In the Perkins' result described above, leading to the extra term (1.8) in the martingale problem, $\gamma = 1$. In this case, the first two terms on the right hand side of (1.11) diverge, and so it is not surprising that the limit processes are singular.

In fact, this is the situation for all $\gamma < \frac{3}{2}$. When $\gamma > \frac{3}{2}$, all the terms on the right hand side of (1.11) vanish in the limit, and, in fact, the limit processes are identical regardless of the prescence or abscence of interaction. When $\gamma = \frac{3}{2}$, however, the situation is rather strange. In this case we have

THEOREM 1.2. *Let $\gamma = \frac{3}{2}$, and b be bounded, continuous, and square integrable on $\Re^{2d}$. Then the sequence (X^μ, W^μ) converges weakly on $D([0,T], M_F(\Re^d) \times (S'_d)^{\otimes d})$ to a limit (X, W). Let Q be the probability measure generated on $D([0,T], M_F(\Re^d) \times (S'_d)^{\otimes d})$ by (X, W), and P that generated by $(\widetilde{X}, \widetilde{W})$. Then the Radon-Nikodym derivative of Q with respect to P is given by $J(\widetilde{X}, \widetilde{W})$, where*

$$
\begin{aligned}
\log\left(J(\widetilde{X}, \widetilde{W})\right) := \ & \int_0^T \int_{\Re^{2d}} b(x,y)\, \widetilde{X}_s(dy)\widetilde{W}(dx,ds) \\
& - \tfrac{1}{2} \int_0^T \int_{\Re^{3d}} \langle b(x,y), b(x,z)\rangle\, \widetilde{X}_s(dx)\widetilde{X}_s(dy)\widetilde{X}_s(dz)\, ds.
\end{aligned}
$$

(1.12)

The following, and longest, section of the paper is devoted to the proofs of Theorems 1.1 and 1.2.

(c) Understanding the main results. We now return to the "curiosity" mentioned in the opening line of the paper. Theorem 1.2 indicates that the distributions of (X, W)

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and $(\widetilde{X}, \widetilde{W})$ are different. That is, the mild interaction introduced at (1.5) with $\gamma = \frac{3}{2}$ *does* have an effect in the limit.

On the other hand, if one compares this with the result of Perkins, with $\gamma = 1$, that led to the additional term (1.8) in the martingale problem for X, it is clear that the distributions of X and $\widetilde{X}$ must be identical. (To see this, at an intuitive level, think of (1.5) with a coefficient of μ^{-1} before the final term, and an extra factor of $\mu^{-1/2}$ absorbed into the function b. Then, as $\mu \to \infty$, the interaction function $\mu^{-1/2}b$ also disappears, so there is no correction term (1.8) for the limiting martingale problem.)

Consequently, while the pair (X, W) does "remember" the interaction, it is only at the level of the joint distribution, and not at the level of the marginal distribution of X. Since in the usual study of superprocesses the process W does not appear anywhere, but nevertheless exists (albeit flittering etherally in the background) and contains useful information, we refer to it as the "ghost process".

A similar analysis to that above, for general dimension d, or results of Barlow and Perkins (1993), for $d \geq 5$, show that this phenomenon can also be carried across to the level of historical Brownian motion. (See Dawson and Perkins 1991, Dynkin 1991a,b, or Perkins 1992 for a definition and details of this process.) The situation there is identical. While W continues to remember the weak interaction, this is not the case for the historical process, which continues to behave as a historical process that arose as the limit of non-interacting Brownian motions. We find this phenomenon particularly interesting, since it provides a specific example of behaviour observable at the level of the particle system that is not only non-reconstructible, but not even noticeable, at the level of the limiting historical process.

Finally, to better understand the relationship between the pairs (X, W) and $(\widetilde{X}, \widetilde{W})$, note that when there is no interaction the latter pair can be characterised as the solution of the martingale problem formed by combining (1.7) (with $\widetilde{X}$ replacing X there) with the requirement that, for every $\psi \in S_d$, $\widetilde{W}_t(\psi)$ is a continuous, square integrable martingale such that $\widetilde{W}_0 = 0$ and

$$(1.13) \qquad \langle \widetilde{W}(\psi) \rangle_t = I_d \times \int_0^t \widetilde{X}_s(\psi^2)\, ds, \quad \text{and} \quad \langle \widetilde{W}(\psi), Z(\phi) \rangle_t = 0,$$

where ϕ and Z are as in (1.7) and I_d is the $d \times d$ identity matrix. (cf. Theorem 2.4 below.) We then have

THEOREM 1.3. *Let* $(\widetilde{X}, \widetilde{W})$ *be the solution of the martingale problem described above. When b satisfies the conditions of Theorem 1.2,*

$$(1.14) \quad (X_t(\phi), W_t(\psi)) \stackrel{\mathcal{L}}{=} \left(\widetilde{X}_t(\phi), \widetilde{W}_t(\psi) + \int_0^t \int_{\mathbb{R}^{2d}} b(x, y)\psi(x)\, \widetilde{X}_s(dx)\widetilde{X}_s(dy)\, ds \right),$$

where equality in law is to be understood as equality between processes on $D([0, \infty), M_F(\mathbb{R}^d) \times (S_d')^{\otimes d})$.

We shall not prove this Theorem, other than to say that it follows directly from the techniques used in Walsh (1986) to establish the SPDE associated with (1.7a) and the SPDE's of his Chapter 8.

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2. PROOFS

We start with the

(a) Proof of Theorem 1.1. The proof proceeds in a number of stages, since the application of the usual Girsanov formula, while not difficult, is not straightforward. The basic difficulty is that because of the branching mechanism the number of particles (diffusions) keeps changing, and the usual Girsanov formula is for a fixed number of diffusions. Thus we start by looking at time intervals of the form $[k/\mu, (k+1)/\mu)$, during which there is no branching, and consider, for fixed k and μ, the following system of diffusions, in which $i = 1, \ldots, d$, and $\alpha, \beta \sim k/\mu$:

$$X_t^{\alpha,i} = X_{\frac{k}{\mu}}^{\alpha,i} + \int_{\frac{k}{\mu}}^t h_s^\alpha \, W^{\alpha,i}(ds) + \mu^{-\gamma} \sum_{\beta \neq \alpha} \int_{\frac{k}{\mu}}^t b^i(X_s^\alpha, X_s^\beta) h_s^\alpha h_s^\beta \, ds,$$

$$\text{(2.1)} \qquad h_t^\alpha = h_{\frac{k}{\mu}}^\alpha,$$

$$N_t^{\alpha-1} = N_{\frac{k}{\mu}}^{\alpha-1},$$

with initial conditions

$$\text{(2.2)} \qquad X_{\frac{k}{\mu}}^{\alpha,i} = X_{\frac{k}{\mu}-}^{\alpha-1,i}, \qquad h_{\frac{k}{\mu}}^\alpha = \tfrac{1}{2} N^{\alpha-1} h_{\frac{k}{\mu}-}^{\alpha-1}, \qquad N_{\frac{k}{\mu}}^{\alpha-1} = N^{\alpha-1},$$

where the N^α and $W^\alpha = (W^{\alpha,1}, \ldots, W^{\alpha,d})$ are as in the previous section. Because of the restrictions on b, this system determines a unique diffusion on the space $E_{k,\mu} : \Re^{2^k dK_\mu} \times \{0,1\}^{2^k K_\mu} \times \{0,2\}^{2^{k-1} K_\mu}$.

Let $\eta_k^\mu(t) := (X_t^\alpha, h_t^\alpha, N_t^{\alpha-1})_{\alpha \sim k/\mu}$ be a solution of (2.1) with the initial conditions (2.2). Then $\eta_k^\mu \in G_k^\mu$, the space of continuous functions on $[k/\mu, (k+1)/\mu)$ with values in $E_{k,\mu}$. Let Q_k^μ denote the measure that $(\widehat{X}_t^\alpha, \widehat{h}_t^\alpha, \widehat{N}_t^{\alpha-1})$ generates on G_k^μ, and P_k^μ the measure corresponding to the case $b = 0$ in (2.1) with the same initial conditions as in (2.2). Write $J_k^\mu = dQ_k^\mu/dP_k^\mu$ for the Radon-Nikodym derivative of Q_k^μ with respect to P_k^μ.

We need one more piece of notation: Let

$$\pi_1(\eta(\cdot)) = \widehat{X}_\cdot^\alpha = \{X_\cdot^\alpha\}_{\alpha \sim \frac{k}{\mu}},$$

$$\pi_2(\eta(\cdot)) = \widehat{h}_\cdot^\alpha = \{h_{k/\mu}^\alpha\}_{\alpha \sim \frac{k}{\mu}},$$

$$\pi_3(\eta(\cdot)) = \widehat{N}_\cdot^{\alpha-1} = \{N^{\alpha-1}\}_{\alpha \sim \frac{k}{\mu}}$$

be the projections of G_μ^k onto the spaces of functions taking values in $\Re^{2^k dK_\mu}$, $\{0,1\}^{2^k K_\mu}$ and $\{0,2\}^{2^{k-1} K_\mu}$, respectively. Now write

$$(\widehat{X}_t^\alpha, \widehat{h}_t^\alpha, \widehat{N}_t^{\alpha-1}) := (\pi_1(\eta_k^\mu(t)), \pi_2(\eta_k^\mu(t)), \pi_3(\eta_k^\mu(t)).$$

As in the previous section, we shall indicate a lack of interaction in the diffusion mechanism (i.e. $b = 0$) with an extra tilde. Then we have

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LEMMA 2.1. *Let b be bounded Lipschitz. Then the Radon-Nikodym derivative J_k^μ defined above exists and is given by*

$$
\begin{aligned}
J_k^\mu\big(\widetilde{\eta}_k^\mu(t)\big) &= J_k^\mu\big(\widehat{\widetilde{X}_t^\alpha, \widehat{h}_t^\alpha, \widehat{N}_t^{\alpha-1}}\big) \\
&= \frac{dQ_k^\mu}{dP_k^\mu}\big(\widehat{\widetilde{X}_t^\alpha, \widehat{h}_t^\alpha, \widehat{N}_t^{\alpha-1}}\big) \\
&= \exp\Big\{ \int_{\frac{k}{\mu}}^{t} \Big[\mu^{1-\gamma} \sum_{\alpha \sim k/\mu} \langle \widetilde{X}_s^\mu(b(\widetilde{X}_s^\alpha, \cdot)) h_s^\alpha, d\widetilde{X}_s^\alpha \rangle \\
&\quad - \mu^{-\gamma} \sum_{\alpha \sim k/\mu} \langle b(\widetilde{X}_s^\alpha, \widetilde{X}_s^\alpha) h_s^\alpha, d\widetilde{X}_s^\alpha \rangle - \tfrac{1}{2}\mu^{3-2\gamma} \widetilde{X}_s^\mu \otimes \widetilde{X}_s^\mu \otimes \widetilde{X}_s^\mu (\langle b(x,y), b(x,z) \rangle) \\
&\quad + \mu^{2-2\gamma} \widetilde{X}_s^\mu \otimes \widetilde{X}_s^\mu (\langle b(x,x), b(x,y) \rangle) - \tfrac{1}{2}\mu^{1-2\gamma} \widetilde{X}_s^\mu (\|b(x,x)\|^2) \Big] \, ds \Big\},
\end{aligned}
$$

where $(X_s^\mu(b))^i = X_s^\mu(b^i)$, $i = 1, \dots, d$, and $X_s^\mu(b(\cdot)) = \{(X_s^\mu(b))^i\}_{i=1}^d = (X_s^\mu(b^i))_{i=1}^d$.

PROOF: The Cameron-Martin-Girsanov change of measure formula immediately gives the existence of J_k^μ and the fact that it is equivalent to

$$
\begin{aligned}
\exp\Big\{ &\int_{\frac{k}{\mu}}^{t} \mu^{-\gamma} \sum_{\substack{\alpha,\beta \sim k\lambda \\ \alpha \neq \beta}} \sum_{i=1}^{d} b^i(\widetilde{X}_s^\alpha, \widetilde{X}_s^\beta) h_s^\alpha h_s^\beta \, X^{\alpha,i}(ds) \\
&- \tfrac{1}{2} \int_{\frac{k}{\mu}}^{t} \mu^{-2\gamma} \sum_{\substack{\alpha,\beta,\delta \\ \beta \neq \alpha, \delta \neq \alpha}} \sum_{i=1}^{d} b^i(\widetilde{X}_s^\alpha, \widetilde{X}_s^\beta) b^i(\widetilde{X}_s^\alpha, \widetilde{X}_s^\delta) h_s^\alpha h_s^\beta h_s^\delta \, ds \Big\}
\end{aligned}
$$

Verifying the lemma now involves no more than rewriting the above in measure notation. The extra terms in the expression in the statement of the lemma come from the fact that while the product measures $X_s^\mu \otimes X_s^\mu$, and $X_s^\mu \otimes X_s^\mu \otimes X_s^\mu$ include diagonal terms, this is not so for the summation above. The extra expressions, therefore, are simply to counterbalance diagonal terms. ∎

We can now start piecing together the result of Lemma 2.1 to get something more general.

LEMMA 2.2. *Take $m \geq 1$ and η_k^μ and $\widetilde{\eta}_k^\mu$, $k = 0, \dots, m$, respectively, solutions of (2.1) and (2.2) with, and without, interaction. Let f_k be bounded Borel measurable functionals on G_k^μ, and let J_k^μ be as in the previous lemma. Then, under the conditions of Lemma 2.1,*

$$
(2.3) \qquad E\Big(\prod_{k=0}^{m} f_k(\eta_k^\mu) \Big) = E\Big(\prod_{k=0}^{m} f_k(\widetilde{\eta}_k^\mu) J_k^\mu(\widetilde{\eta}_k^\mu) \Big).
$$

PROOF: We want to compare the solutions of (2.1) with and without interaction, but cannot apply a Girsanov change of measure directly because η_k^μ and $\widetilde{\eta}_k^\mu$ have different initial conditions. We get around this by chain conditioning.

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Let $\mathcal{F}_{\frac{k}{\mu}} := \sigma\{X_t^\alpha, h_t^\alpha, N_t^{\alpha-1} : \alpha \sim t,\ 0 \le t \le k/\mu\}$, and take two bounded functionals f_k, f_{k+1} as in the statement of the lemma. Then

$$E\Big[f_k(\eta_k^\mu)\,f_{k+1}(\eta_{k+1}^\mu)\Big|\mathcal{F}_{\frac{k}{\mu}}\Big]$$

$$= E\Big[E\big(f_k(\eta_k^\mu)\,f_{k+1}(\eta_{k+1}^\mu)\big|\mathcal{F}_{\frac{k+1}{\mu}}\big)\Big|\mathcal{F}_{\frac{k}{\mu}}\Big]$$

$$= E\Big[f_k(\eta_k^\mu)\,E\big(f_{k+1}(\widetilde{\eta}_{k+1}^\mu)\,J_n^{k+1}(\widetilde{\eta}_{k+1}^\mu)\big|\mathcal{F}_{\frac{k+1}{\mu}}\big)\Big|\mathcal{F}_{\frac{k}{\mu}}\Big]$$

$$= E\Big[f_k(\eta_k^\mu)\,E\big(f_{k+1}(\widehat{\widetilde{X}_{\frac{k}{\mu}}^\alpha + \widehat{h}^\alpha \oplus \widehat{W}^\alpha}, \widehat{h}^\alpha, \widehat{N}^{\alpha-1})$$

$$\times\ J_{k+1}^\mu\big(\widehat{\widetilde{X}_{\frac{k}{\mu}}^\alpha} + \widehat{h}^\alpha \oplus \widehat{W}^\alpha, \widehat{h}^\alpha, \widehat{N}^{\alpha-1}\big)\big|\mathcal{F}_{\frac{k+1}{\mu}}\big)\Big|\mathcal{F}_{\frac{k}{\mu}}\Big].$$

Note that the arguments of both f_{k+1} and J_{k+1}^μ can be written as

$$\Big(\pi_1\big(\eta_k^\mu\big(\big(\tfrac{k+1}{\mu}\big)-\big)\big)\ +\ \pi_2\big(\eta_k^\mu\big(\big(\tfrac{k+1}{\mu}\big)-\big)\big) \oplus \tfrac{1}{2}\widehat{N}^{\alpha-1} \oplus \widehat{W}^\alpha,$$

$$\pi_2\big(\eta_k^\mu\big(\big(\tfrac{k+1}{\mu}\big)-\big)\big) \oplus \tfrac{1}{2}\widehat{N}^{\alpha-1},\quad \widehat{N}^{\alpha-1}\Big),$$

where $\widehat{W}^\alpha := \{W^\alpha\}_{\alpha \sim k/\mu}$, and $\oplus$ indicates an appropriate composition. (e.g. $\widehat{h}_t^\alpha \oplus \widehat{W}_t^\alpha = \{(h_t^\alpha W_t^{\alpha,i})_{i=1}^d\}_{\alpha \sim t}$.) That is, the entire expectation here can be written in the form

$$E\Big[\varphi_k(\eta_k^\mu, \widehat{N}^{\alpha-1}, \widehat{W}^\alpha)\Big|\mathcal{F}_{\frac{k}{\mu}}\Big]$$

for some function φ_k. Since $(\widehat{N}^{\alpha-1}, \widehat{W}^\alpha)$ is independent of $\mathcal{F}_{k/\mu}$, this is equal to an expression of the form

$$E\Big[\varphi_k(\widetilde{\eta}_k^\mu, \widehat{N}^{\alpha-1}, \widehat{W}^\alpha) \cdot J_k^\mu(\widetilde{\eta}_k^\mu)\Big|\mathcal{F}_{\frac{k}{\mu}}\Big].$$

Substituting now the explicit form of φ_k, we immediately obtain that

$$E\Big[f_k(\eta_k^\mu)\,f_{k+1}(\eta_{k+1}^\mu)\Big|\mathcal{F}_{\frac{k}{\mu}}\Big]\ =\ E\Big[f_k(\widetilde{\eta}_k^\mu)J_k^\mu(\widetilde{\eta}_k^\mu) \cdot f_{k+1}(\widetilde{\eta}_{k+1}^\mu)J_{k+1}^\mu(\widetilde{\eta}_{k+1}^\mu)\Big|\mathcal{F}_{\frac{k}{\mu}}\Big].$$

Continuing inductively, and removing the conditioning, we obtain (2.3) and the proof of the lemma. ∎

Before we can complete the proof of Theorem 1.1, we need one technical lemma, whose proof follows from the techniques in Jakubowski (1986).

LEMMA 2.3. *The collection of the functions of the form*

$$\exp\{-\nu_s(\phi) - i\langle \eta_s(\psi), \alpha\rangle\},$$

with ϕ positive and bounded, $\psi \in \mathcal{S}_d$, $\alpha \in \Re^d$, $s \in [0, T]$, and $(\nu, \eta) \in D([0, T], M_F(\dot{\Re}^d) \times (\mathcal{S}_d')^{\otimes d})$, generates the Borel σ-algebra on $D([0, T], M_F(\dot{\Re}^d) \times (\mathcal{S}_d')^{\otimes d})$. ($\dot{\Re}^d$ is the one point compactification of $\Re^d$.)

We can now turn to the

PROOF OF THEOREM 1.1: There are two, now simple, steps to the proof.

Firstly, we note, from Lemma 2.1, that if $t = (m+1)/\mu$, then the product $\prod_{k=0}^m J_k^\mu(\widetilde{\eta}_k^\mu)$ is precisely the expression on the right hand side of (1.11).

Then applying Lemma 2.2 to functionals f_k such that

$$\prod_{k=0}^m f_k(\eta_k^\mu) = \exp\left\{-X_s^\mu(\phi) - i\langle W_s^\mu(\psi), \alpha\rangle\right\},$$

we obtain that

$$E\left\{\exp\left\{-X_s^\mu(\phi) - i\langle W_s^\mu(\psi), \alpha\rangle\right\}\right\} = E\left\{J(\widetilde{X}^\mu, \widetilde{W}^\mu) \cdot \exp\left\{-X_s^\mu(\phi) - i\langle \widetilde{W}_s^\mu(\psi), \alpha\rangle\right\}\right\},$$

where J is the Radon-Nikodym derivative of Theorem 1.1 with $t = T$. Lemma 2.3, the fact that the $\widetilde{X}^\mu$ take values in $M_F(\Re^d)$ with probability one, and a standard monotone class argument now finish the proof. ∎

(b) Convergence of the Radon-Nikodym derivatives. In order to prove Theorem 1.2 from Theorem 1.1, we have, basically, to show three things. The first is that the sequence (X^μ, W^μ) converges weakly to a well defined limit. The second is that the Radon-Nikodym derivatives J^μ of (1.11) for the finite particle system converge weakly to the limit J of (1.12). We shall formulate this carefully as Theorem 2.5 below. The third is to show that the limit of the Radon-Nikodym derivatives is, in fact, the Radon-Nikodym derivative for the limit processes. In this subsection we tackle the second of these problems, which turns out to be prerequisite for the other two.

The basic result behind this section is the following:

THEOREM 2.4. *Let $(\widetilde{X}, \widetilde{W})$ be the unique solution of the martingale problem described in the paragraph preceeding Theorem 1.3. Then*

$$(2.4) \qquad\qquad (\widetilde{X}^\mu, \widetilde{W}^\mu) \Rightarrow (\widetilde{X}, \widetilde{W}) \qquad as\ \mu \to \infty,$$

in $D([0, \infty), M_F(\Re^d) \times (S_d')^{\otimes d})$.

OUTLINE OF PROOF: The convergence of each of the individual components of (2.4) is well known. (cf. Walsh 1986, for example.) What is new here is the joint convergence on the product space.

A similar example is worked out in detail in Mytnik and Adler (1993), so we shall not repeat the details here. We simply note that there are, as usual, two parts to the proof. The first establishes the fact that all limit points of $(\widetilde{X}^\mu, \widetilde{W}^\mu)$ satisfy the appropriate limiting martingale problem, and that the solution of this martingale problem is unique. An appropriate core of test functions for this part of the proof is the family of functions of the form described in Lemma 2.3

Tightness for the pair, given tightness for each component in (2.4), then follows from an application of Theorem 4.6 of Jakobowski (1986). ∎

On comparing (1.11) and (1.12), it is clear that as far as the weak convergence of J^μ to J is concerned, it suffices to establish that the last three expressions on the right hand side of (1.11) converge to 0 in L^2 as $\mu \to \infty$, along with the joint weak convergences

$$(2.5) \qquad \int_0^T \int_{\Re^{2d}} b(x,y)\,\widetilde{X}_s^\mu(dy)\widetilde{W}^\mu(dx,ds) \;\Rightarrow\; \int_0^T \int_{\Re^{2d}} b(x,y)\,\widetilde{X}_s(dy)\widetilde{W}(dx,ds),$$

and

$$\int_0^T \int_{\Re^{3d}} \langle b(x,y), b(x,z)\rangle\, \widetilde{X}_s^\mu(dx)\widetilde{X}_s^\mu(dy)\widetilde{X}_s^\mu(dz)\,ds$$

$$(2.6) \qquad\qquad \Rightarrow \int_0^T \int_{\Re^{3d}} \langle b(x,y), b(x,z)\rangle\, \widetilde{X}_s(dx)\widetilde{X}_s(dy)\widetilde{X}_s(dz)\,ds,$$

(remember that $\gamma = \frac{3}{2}$). Given (2.4), (2.5) and (2.6) seem quite reasonable.

We start with the convergence to zero of the first of these three terms of (1.11). Note first that (1.13) is true with $\widetilde{W}$ and $\widetilde{X}$ replaced, respectively, by $\widetilde{W}^\mu$ and $\widetilde{X}^\mu$, (cf. Walsh 1986) so that

$$E\Big[\mu^{-1} \int_0^T \int_{\Re^d} \langle b(x,x), \widetilde{W}^\mu(dx,ds)\rangle\Big]^2 = \mu^{-2} E\Big[\Big\langle \int_0^\cdot \int_{\Re^d} \langle b(x,x), \widetilde{W}^\mu(dx,ds)\rangle\Big\rangle_T\Big]$$

$$= \mu^{-2} \sum_{i=1}^d E \int_0^T \int_{\Re^d} (b^i(x,x))^2\, \widetilde{X}_s^\mu(dx)\,ds$$

$$\leq \mu^{-2} \sum_{i=1}^d \sup_{x \in \Re^d} (b^i(x,x))^2 TE\big(\sup_{s \leq T} \widetilde{X}_s^\mu(1)\big)$$

$$\to 0 \qquad \text{as } \mu \to \infty,$$

since $\sup_\mu E(\sup_{s \leq T} \widetilde{X}_s^\mu(1)) < \infty$.

The other two terms are handled similarly, with in both cases the convergence to 0 coming from a negative power of μ, the boundedness of b, and the uniform boundedness of the moments of $\widetilde{X}^\mu(1)$.

We now turn to (2.5) and (2.6). Both can be handled similarly, and we shall look only at (2.5), which, since it involves both $\widetilde{W}$ and $\widetilde{X}$ is somewhat more delicate. (Note that (2.6) also follows from results of Feldman and Iyer 1993.)

Now, for the first time, we use the assumption that $b \in L^2(\Re^d \times \Re^d)$. For $n \geq 1$ let

$$b^n(x,y) := \Big(\sum_{k=1}^n \phi_k^i(x)\psi_k^i(y)\Big)_{i=1}^d,$$

be an approximation to b, such that $\phi_k^i \in S_d$ and $\psi_k^i \in C_l^2(\Re^d)$ (the space of C^2 functions on $\Re^d$ with limits at infinity) for all $k \geq 0$, and such that

$$\int_{\Re^d} \int_{\Re^d} \|b(x,y) - b^n(x,y)\|^2\, dx\,dy \to 0 \qquad \text{as } n \to \infty.$$

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Then, by Kurtz and Protter (1992),

$$Y_{\mu,n} := \int_0^T \int_{\Re^{2d}} b^n(x,y)\, \widetilde{X}_s^\mu(dy)\widetilde{W}^\mu(dx,ds)$$

$$= \sum_{k=1}^n \int_0^T \widetilde{X}_s^\mu(\psi_k) \cdot d\widetilde{W}_s^\mu(\phi_k)$$

$$\Rightarrow \sum_{k=1}^n \int_0^T \widetilde{X}_s(\psi_k) \cdot d\widetilde{W}_s(\phi_k)$$

$$= \int_0^T \int_{\Re^{2d}} b^n(x,y)\, \widetilde{X}_s(dy)\widetilde{W}(dx,ds)$$

$$:= Y_n$$

Let $Y_\mu\ (= Y_{\mu,\infty})$ and $Y\ (= Y_\infty)$ be the same as $Y_{\mu,n}$ and Y_n, respectively, but with b replacing b_n. Using Dynkin's (1988) diagrams to calculate moments of superprocesses, and the particle system versions of these in Adler (1993), it is then not difficult to show that

$$\lim_{n\to\infty} \limsup_{\mu\to\infty} E|Y_{\mu,n} - Y_n| = 0 = \lim_{n\to\infty} E|Y_n - Y|.$$

(cf. similar calculations in Adler 1993 and Adler and Lewin 1993.)

This clearly establishes (2.5). Arguing similarly for (2.6) finally yields

THEOREM 2.5. Let $J^\mu(\widetilde{X}^\mu, \widetilde{W}^\mu)$ be the Radon-Nikodym derivative given by (1.11) with $\gamma = \frac{3}{2}$. Let $b(x,y)\colon \Re^{2d} \to \Re^d$ be a bounded continuous, L^2 function. Then, for each $T > 0$,

$$(2.7) \qquad J^\mu(\widetilde{X}^\mu, \widetilde{W}^\mu) \Rightarrow J(\widetilde{X}, \widetilde{W}) \qquad as\ \mu \to \infty,$$

in $\Re$, where $(\widetilde{X}, \widetilde{W})$ is the process of the preceeding theorem.

(c) The limit. In this subsection we complete the proof of Theorem 1.2 by showing that the pair (X^μ, W^μ) converges on $D([0,T], M_F(\Re^d) \times (S_d')^{\otimes d})$, and that J really is the Radon-Nikodym derivative we claim it to be.

In fact, both these results will follow in a straightforward fashion from the following result, which we shall prove in a moment.

LEMMA 2.6. The sequence $\{J^\mu(\widetilde{X}^\mu, \widetilde{W}^\mu)\}_{\mu=1}^\infty$ is uniformly integrable.

To see why this is enough, take $p, r \geq 1$ and $f\colon D([0,\infty), \Re^{p+rd}) \to \Re$ continuous and bounded. Then, if Lemma 2.6 is true,

$$E\{f(X^\mu, W^\mu)\} = E\{J^\mu(\widetilde{X}^\mu, \widetilde{W}^\mu) \cdot f(\widetilde{X}^\mu, \widetilde{W}^\mu)\}$$

$$(2.8) \qquad\qquad\qquad \to E\{J(\widetilde{X}, \widetilde{W}) \cdot f(\widetilde{X}, \widetilde{W})\}$$

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We can exploit (2.8) to prove both of the opening claims of this subsection, and so complete the proof of Theorem 1.2.

Define a canonical random process (X, W) on $D([0, \infty), M_F(\dot{\Re}^d) \times (S_d')^{\otimes d})$ such that for all bounded measurable functions g

$$(2.9) \qquad E\{g(X, W)\} = E\{J(\widetilde{X}, \widetilde{W}) \cdot g(\widetilde{X}, \widetilde{W})\}.$$

Then, by (2.8),

$$E\{f(X^\mu, W^\mu)\} \to E\{f(X, W)\} \qquad \text{as } \mu \to \infty.$$

The full weak convergence of (X^μ, W^μ) to (X, W), on $D([0, \infty), M_F(\dot{\Re}^d) \times (S_d')^{\otimes d})$ now follows.

Finally, it will follow from the calculations below that the marginal distribution of X is the same with or without interaction, so that the sample paths of X are supported on $M_F(\Re^d)$. From this follows both the weak convergence of (X^μ, W^μ) to (X, W) on $D([0, \infty), M_F(\Re^d), \times (S_d')^{\otimes d})$, and the fact that J is precisely the Radon-Nikodym it is claimed to be.

All that remains therefore is the

PROOF OF LEMMA 2.6: By Proposition 1.1 of Chung and Williams (1990) it suffices to show that

$$(2.10) \qquad E\{J(\widetilde{X}, \widetilde{W})\} = 1.$$

In fact, we shall show something a little stronger, from which also follows our claim above that the distribution of X is the same with or without interaction. Our claim is that

$$
\begin{aligned}
1 = E\Big\{ &\exp\Big[\int_0^T \int_{\Re^d} \langle \widetilde{X}_s(b(x, \cdot)), \widetilde{W}(dx, ds) \rangle \\
(2.11) \qquad &- \tfrac{1}{2} \int_0^T \int_{\Re^{3d}} \langle b(x, y), b(x, z) \rangle \widetilde{X}_s(dx) \widetilde{X}_s(dy) \widetilde{X}_s(dz) \, ds \Big] \Big| \sigma\{\widetilde{X}_s, s \le T\} \Big\}
\end{aligned}
$$

Denote the martingale problem on $M_F(\Re^d) \times \Re^d$ described in the paragraph preceding Theorem 1.3 by $(M, \delta_m \times \delta_0)$, and let $(\widetilde{X}., \widetilde{W}.(\psi))$ be a solution of it, for some fixed $\psi \in S_d^{\otimes d}$. Then $\widetilde{X}$ is also the unique solution of the martingale problem (1.7), and there exists a Brownian motion $B = B^\psi$ on $\Re^d$, independent of $\mathcal{J} := \sigma\{X_s, s \ge 0\}$ (on an extended probability space, if necessary), such that

$$(2.12) \qquad \widetilde{W}_t(\psi) \overset{\mathcal{L}}{=} Y_t := \int_0^t \sqrt{\widetilde{X}_s(\psi^2)} \, dB_s^\psi,$$

where the equivalence in law is for random functions on $D([0, T], \Re^d)$.

To prove this, we need only check that $(\widetilde{X}, Y)$ satisfies the martingale problem $(M, \delta_m \times \delta_0)$:

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(i) Y_t is obviously a continuous martingale with respect to the filtration $\sigma\{X_s, B_s^\psi, s \leq t\}$, and, consequently, also with respect to the filtration $\widetilde{\mathcal{F}}_t := \sigma\{X_s, Y_s, s \leq t\}$. It is also obvious that

$$\langle Y \rangle_t = I_d \times \int_0^t \widetilde{X}_s(\psi^2)\, ds.$$

(ii) As far as the cross covariation is concerned, we claim that, for all $\phi \in C_l^2(\Re^d)$, $\langle Y, Z(\phi) \rangle_t \equiv 0$, since

$$E\{Y_t Z_t(\phi) - Y_s Z_s(\phi) | \widetilde{\mathcal{F}}_s\} = E\{Z_t(\phi)[Y_t - Y_s] | \widetilde{\mathcal{F}}_s\} + Y_s E\{Z_t(\phi) - Z_s(\phi) | \widetilde{\mathcal{F}}_s\}$$

$$= E\Big\{Z_t(\phi) E\Big[\int_s^t \sqrt{\widetilde{X}_u(\psi^2)}\, dB_u^\psi \Big| \widetilde{\mathcal{F}}_s \vee \mathcal{J}\Big] \Big| \widetilde{\mathcal{F}}_s\Big\},$$

since $Z_t(\phi)$ is a $\mathcal{F}_t^X$ martingale. However, since B^ψ is independent of X, it follows that the inner expectation above is identically zero, and so we have that $\langle Y, Z(\phi) \rangle_t \equiv 0$, as claimed.

The equivalence (2.12) now follows from the uniqueness of the solution of the martingale problem $(M, \delta_m \times \delta_0)$.

Given this equivalence, and forming an exponential martingale, it is now immediate that, for $\phi^i \in C_l^2(\Re^d)$, $\psi^i \in \mathcal{S}_d$,

$$E\Big\{ \exp\Big[\int_0^T \langle \widetilde{X}_s(\phi), \widetilde{W}(\psi, ds) \rangle - \tfrac{1}{2} \sum_{i=1}^d \int_0^T \big(\widetilde{X}_s(\phi^i)\big)^2 \cdot \widetilde{X}_s((\psi^i)^2)\, ds \Big] \Big| \mathcal{J} \Big\} = {}_{\bot}.$$

This is precisely (2.11), for the case $b(x, y) = (\phi^i(x)\psi^i(y))_{i=1}^d$. The extension to $b(x, y) = b^n(x, y) = (\sum_{k=1}^n \phi_k^i(x)\psi_k^i(y))_{i=1}^d$ follows from a multi-dimensional version of the same argument. Choosing the ϕ_k and ψ_k such that $b_n \overset{L^2}{\to} b$ as $n \to \infty$, all the while conditioning on $\mathcal{J}$, establishes (2.11) in general, and so completes the proof of Theorem 2.2. ∎

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